THE CONSTRUCTIONS OF THE SMALLEST PRIME IDEAL FROM A GIVEN SET OF A NOETHERIAN RING

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1. Introduction

In ring theory, for some purpose, we want to find the smallest prime ideal which contains a given subset A of a ring R. It is theoretically not difficult to show that there exists an ideal which is the smallest ideal containing A. However, there does not necessarily exist such a smallest prime ideal containing A. This is because the intersection of two prime ideals is not always a prime ideal. Suppose, under certain conditions, there does exist such a smallest prime ideal. Let F be the collection of all prime ideals of R that contains A. Then it is easy to prove that the intersection $\cap F$ is the required smallest prime ideal.

If we were to attempt to find the smallest prime ideal containing A in this way, we would need to find all the prime ideals of R that contain A, and then find their intersection. This is not an easy task in any but the simplest cases. The purpose of this paper is, under certain restrictions on A, to furnish some alternative methods for the construction of such prime ideal.

2. preliminaries

We shall collect and derive in this section certain results of elementary nature of ideals and Noetherian rings which will be needed in our purpose. For the sake of convenience "ring", in this paper, will always mean a commutative ring with a unity element.

Definition 1. A subset N of a ring R is an ideal if N is an additive subgroup of R such that $ra \in N$ for all $a \in N$ and all $r \in R$.

Theorem 1. A subset N is an ideal of R iff N satisfies the following conditions:

(1) If a, $b \in N$, then $a-b \in N$,

(2) If $a \in N$, then $ra \in N$ for all $r \in R$.

Proof: See [1] Theorem 3, Page 9; [2] Page 4.

Theorem 2. Let A be an arbitrary subset of R. Then the set

$$(A)=\{x \mid x=r_1a_1+r_2a_2+\cdots+r_sa_s,\ r_i \in R,\ a_i \in A,\ s \in Z\ (Z \text{ the positive integers})\}$$

forms an ideal and (A) is the smallest ideal containing A.

Proof: We shall first show that (A) is an ideal. Let x, $y \in (A)$. Then

$$x=x_1a_1+\cdots\cdots+x_sa_s,$$

$$y=y_1b_1+\cdots\cdots+y_tb_t,$$

where $x_i, y_i \in R$; a_i, $b_i \in A$. Now

$$x-y=(x_1a_1+\cdots\cdots+x_sa_s)-(y_1b_1+\cdots\cdots+y_tb_t)$$

$$=x_1a_1+\cdots\cdots+x_sa_s+(-y_1)b_1+\cdots\cdots+(-y_t)b_t$$

and $$rx=(rx_1)a_1+\cdots\cdots+(rx_s)a_s.$$

Hence $x-y\in(A)$, $rx\in(A)$. By Theorem 1, (A) is an ideal. It is clear that $A\subseteq(A)$, and therefore the theorem will be established if we show that $(A)\subseteq N$ for any ideal N of R that contains A. Suppose $x\in(A)$. Then $x=x_1a_1+\cdots\cdots+x_sa_s$. But $a_i\in A\subseteq N$ and N is an ideal, hence $x\in N$. This means that $(A)\subseteq N$.

Definition 2. An ideal N in R is a prime ideal if $ab\in N$ implies either $a\in N$ or $b\in N$.

Definition 3. A subset S of R is multiplicatively closed if $a,b\in S$ implies $ab\in S$.

Theorem 3. An ideal N of R is a prime ideal iff $C(N)$, the complement of N on R, is multiplicatively closed.

Proof: The proof of this theorem follows immediately from the Definition 3 and Definition 2.

Definition 4. Let N be an ideal of R. Then the set

$$\{x\mid x^m\in N \text{ for some positive integer } m\}$$

is called the radical of N, and is denoted by $[N]$.

Theorem 4. The radical $[N]$ of an ideal N is an ideal.

Proof: If $x,y\in[N]$, then $x^m,y^n\in N$ for some positive integers m, n. Now $(x-y)^{m+n}$ can be written as a sum of terms $x^u(-y)^v$ with $u\geq 0$, $v\geq 0$, and $u+v=m+n$. Consequently we have either $u\geq m$ or $v\geq n$. In either case, by Definition 1, we have $x^u(-y)^v\in N$, and so that $(x-y)^{m+n}\in N$. This means that $(x-y)\in[N]$. Also $(rx)^m=r^mx^m\in N$ and therefore $rx\in[N]$. Hence $[N]$ is an ideal by Theorem 1.

Definition 5. An ideal N is called a primary ideal of R if $ab\in N$ and $a\notin N$ implies $b^m\in N$ for some positive integer m.

Theorem 5. If N is a primary ideal of R, then $[N]$ is a prime ideal which is the smallest prime ideal containing N.

Proof: See [2] proposition 3, P.10; [3] P. 5.

Definition 6. A ring R is called Noetherian if every ideal N of R is finitely generated.

Theorem 6. R is Noetherian ring iff for any non-empty collection H of ideals in R, there is always an ideal $M\in H$ such that if $B\in H$ and $B\supseteq M$, then $B=M$.

This is one of the alternative definitions of the Noetherian ring which is useful in proving the fundamental properties (Theorem 7 below) of the Noetherian ring. See [1] P.115; [2] P.20.

Definition 7. If an ideal N can be expressed in

$$N=Q_1\cap Q_2\cap\cdots\cdots\cap Q_n$$

where Qi are primary ideals, we say that N has a primary decomposition.

Theorem 7. Every ideal N of a Noetherian ring R has a primary decomposition.

Proof: See [2] P. 21.

3. Constructions of prime ideals

Observing the constructions between the set A and the ideal (A), and that between the primary ideal N and the radical $[N]$ as shown in section 2, we might find the suggestion to furnish several equivalent ways to construct the smallest prime ideal from a given subset A. We now state them as the following theorem.

THEOREM. Let A be a subset of a Noetherian ring R. Then the following set constructions are equivalent:

(1) $\cap H = \cap \{P \in H \mid H$ is the collection of all prime ideals of R containing $A\}$;

(2) $[(A)] = \{x \in R \mid x^m = x_1a_1 + \cdots\cdots + x_sa_s,\ x_i \in R,\ a_i \in A,\ m, s$ are some positive integers$\}$;

(3) $(\overline{A}) = \{x \in R \mid M \cap (A) \neq \phi$ for every multiplicatively closed set $M \subseteq R$ containing $x\}$.

Moreover, if (A) is a primary ideal of R, then each of these sets is the smallest prime ideal of R containing A.

Proof: First, we shall show that $[(A)] \subseteq (\overline{A})$. Let $x \in [(A)]$. Then there exist two positive integers m and s such that

$$x^m = x_1a_1 + \cdots + x_sa_s \in (A).$$

To show that $x \in (\overline{A})$, since R itself is a multiplicatively closed set, it is sufficient to consider those multiplicatively closed subsets which contain x, and prove that each of them has a nonempty intersection with (A). Now let M be any multiplicatively closed subset with $x \in M$. Then $x^m \in M$, and therefore $M \cap (A) \neq \phi$. Hence we have $[(A)] \subseteq (\overline{A})$.

Next we shall show that $(\overline{A}) \subseteq \cap H$. Let us observe that A and $(\overline{A})$ are contained in precisely the same prime ideals. To prove this, it is sufficient to consider the sets (A) and $(\overline{A})$, since, by Theorem 2, (A) is the smallest ideal containing A. By the construction of $(\overline{A})$, it is clear that $(A) \subseteq (\overline{A})$, so any prime ideal which contains $(\overline{A})$ necessarily contains (A). Now suppose that P is a prime ideal of R such that $(A) \subseteq P$, and let $a \in (\overline{A})$. If $a \notin P$, then $C(P)$ would be a multiplicatively closed set containing a. Thus $C(P) \cap (A) \neq \phi$. On the other hand, since $(A) \subseteq P$, we have $C(P) \cap (A) = \phi$. This contradiction shows that $a \in P$, and hence $(\overline{A}) \subseteq P$. Also $P \in$ H. Thus $(\overline{A}) \subseteq \cap H$.

We have seen that

$$[(A)] \subseteq (\overline{A}) \subseteq \cap H,$$

and we shall complete the proof by showing that $\cap H \subseteq [(A)]$. Let us look back to section 2, Theorem 2 shows that (A) is the smallest ideal containing A. Since R is a Noetherian ring, then from Theorem 7 we see that (A) has a primary decomposition

$$(A) = Q_1 \cap Q_2 \cap \cdots\cdots \cap Q_n,$$

where Q_i are primary ideals of R. Now

$$[(A)] = [Q_1 \cap Q_2 \cap \cdots\cdots \cap Q_n]$$
$$= [Q_1] \cap [Q_2] \cdots\cdots \cap [Q_n].$$

By Theorem 5, $[Q_i] \in H$, for $i = 1, 2, \cdots\cdots, n$. Hence

$$\cap H \subseteq [(A)].$$

Moreover, if (A) is a primary ideal, then, by Theorem 5 again, $[(A)]$ is a prime ideal and is the smallest prime ideal which contains A.

COROLLARY. Let A be a subset of a Noetherian R. Then the necessary and sufficient condition for the existence of the smallest prime ideal containing A is that the set (A) is a primary ideal.

Proof: The sufficiency is a consequence of the theorem.

To prove the necessity, we assume that there exists a smallest prime ideal containing A. From our theorem above, it must be $[(A)]$. Now, if $ab \in (A)$, then $ab \in [(A)]$. By the fact that $[(A)]$ is a prime ideal, we know that either $a \in [(A)]$ or $b \in [(A)]$. Moreover, since $[(A)]$ is the radical ideal of (A), the above argument means that either $a^m \in (A)$ or $b^n \in (A)$. Hence (A) is a primary ideal.

4. Examples

Example 1. The set

$$R=\{0,\ 1,\cdots\cdots 10,\ 11\},$$

under addition module 12, and multiplication module 12, forms a Noetherian ring.

(a) Consider the subset $A=\{4,\ 8\}$. Then

$$(A)=\{r_1 4+r_2 8=4(r_1+2r_2)\,|\,r_1,\ r_2\in R\}$$
$$=\{0,\ 4,\ 8\}$$

Evidently, (A) is a primary ideal. Now

(1) There are only two prime ideals containing A, namely,

R and $B=\{0,\ 2,\ 4,\ 6,\ 8,\ 10\}$.

Hence $\cap \mathrm{H}=\mathrm{B}=\{0,\ 2,\ 4,\ 6,\ 8,\ 10\}$.

(2) $[(A)]=\{x\,|\,x^m\in(A)$ for some integer $m\}$

$$=\{0,\ 2,\ 4,\ 6,\ 8,\ 10\}.$$

(3) Since $(A)\subseteq(\overline{A})$, we have $0,\ 4,\ 8\in(\overline{A})$. Now we examine all other elements of R. It is clear that $\{1,\ 5\}$, $\{1,\ 7\}$, $\{1,\ 11\}$, and $\{3,9\}$ are those multiplicatively closed sets of R which do not meet (A). Accordingly $1,\ 3,\ 5,\ 7,\ 9,\ 11\notin(\overline{A})$. To show the elements $2,\ 6,\ 10\in(\overline{A})$ is sufficiently to find the smallest multiplicatively closed sets which contain 2, 6, 10; and meet (A) respectively. Obviously, the sets $\{2,\ 4,\ 8\}$, $\{0,6,\}$, and $\{4,\ 10\}$ will do.

Hence

$$(\overline{A})=\{0,\ 2,\ 4,\ 6,\ 8,\ 10\}.$$

We have shown that

$$\cap \mathrm{H}=[(A)]=(\overline{A})=\{0,\ 2,\ 4,\ 6,\ 8,\ 10\},$$

and it is really the smallest prime ideal containing $\{4,\ 8\}$.

(b) Consider the subset $\{6\}$. Then $(A)=\{0,\ 6\}$. Look at $2\times 3\in(A)$, but $2^m\notin(A)$, $3^m\notin(A)$ for all positive integer m. Thus (A) is not a primary ideal. We can show that

$$\{0,\ 3,\ 6,\ 9\}\in H,\ \{0,\ 2,\ 4,\ 6,\ 8,\ 10\}\in H.$$

Hence $\cap H=\{0,6,\}$. Evidently $\{0,\ 6\}$ is not a prime ideal.

From the construction point of view, the approach to $\cap H$ by $[(A)]$ and $(\overline{A})$ may seem to be not very great improvement over $\cap H$ itself; but it often supplies a good deal of insight into the character of $\cap H$. This is illustrated by the following example.

Example 2. Let R be a Noetherian ring. Then, by Hilbert's basis Theorem, the polynomial ring $R[x]$ is also a Noetherian ring.

Consider the subset $A=\{x^2\}$ of $R[x]$. In order to construct $\cap H$, we must find all prime ideals containing A. This is somewhat difficult. But if we start from (A), then

$$(A)=\{x^2 f(x)\,|\,f(x)\in R[x]\},$$

and

$$[(A)]=\{xf(x)\,|\,f(x)\in R[x]\}.$$

REFERENCES

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〔2〕 D. G. Northcott (1953). Ideal Theory.
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諾氏環中最小質理想子環之構造

解 萬 臣

在環論 (Ring Theory) 中常需在某一環上求其含已知集合之質理想子環 (Prime ideal)。但二質理想子環之交截集合並非恒爲一質理想子環。本文提出在諾氏環 (Noetherian Ring) 上幾種相互等效之集合構造法，且在某種條件下，此等方法求得之集合均爲所需之最小質理想子環。並且由此求得此種最小質理想子環存在之必要而充分條件。

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In the ring theory, we want to find the smallest prime ideal which contains a given subset of a ring. The intersection of two prime ideals is not always a prime ideal. The purpose of this paper is, under certain restrictions, to furnish some alternative methods for the construction of such prime ideal. From these constructions, we can find a necessary and sufficient condition for the existence of such smallest prime ideal.